

Differentiation

1

$$f(x) = \sin(x)$$

$$f'(x) = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x}$$

$$= \lim_{t \rightarrow x} \frac{\sin(t) - \sin(x)}{t - x}$$

$$= \lim_{t \rightarrow x} \cancel{\cos\left(\frac{t+x}{2}\right)} \sin\left(\frac{t-x}{2}\right)$$

$$= \lim_{t \rightarrow x} \cos\left(\frac{t+x}{2}\right)$$

$$= \cos(x)$$

Differentiation

2

$$f(x) = \cos(x)$$

$$f'(x) = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} = \lim_{t \rightarrow x} \frac{\cos(t) - \cos(x)}{t - x}$$

$$= \lim_{t \rightarrow x} \frac{-\cancel{2} \sin\left(\frac{t+x}{2}\right) \sin\left(\frac{t-x}{2}\right)}{\frac{t-x}{2} \lim_{=1}}$$

$$= \lim_{t \rightarrow x} -\sin\left(\frac{t+x}{2}\right) \rightarrow \underline{\underline{-\sin(x)}} \blacksquare$$

Differentiation

3

$$f(x) = e^x$$

$$f'(x) = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} = \lim_{t \rightarrow x} \frac{e^t - e^x}{t - x}$$

Substitute $t = x + h$.

$$= \dots \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h}$$

$$= e^x \left(\lim_{h \rightarrow 0} \frac{e^h - 1}{h} \right) = \underline{\underline{e^x}} \quad \blacksquare$$

Differentiation

4

$$f(x) = \ln(x)$$

$$f'(x) = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} = \lim_{t \rightarrow x} \frac{\ln(t) - \ln(x)}{t - x}$$

Substituting $t = x + hx$

$$= \dots \lim_{h \rightarrow 0} \frac{\ln(x + hx) - \ln(x)}{hx}$$

$$= \frac{1}{x} \left(\lim_{h \rightarrow 0} \frac{\ln(1+h)}{h} \right) \quad \text{---} \quad 1$$

$$= \underline{\underline{\frac{1}{x}}} \quad \forall x > 0 \quad \blacksquare$$

Differentiation

5

$$f(x) = -\sin(x) + e^x - 1$$

$$f'(x) = -\cos x + e^x$$

$$g(x) = \cos(x) - 1$$

$$g'(x) = -\sin(x)$$

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)}$$

$$= \lim_{x \rightarrow 0} \frac{-\cos(x) + e^x}{-\sin(x)} = \lim_{x \rightarrow 0} \frac{\sin(x) + e^x}{-\cos(x)} \quad [11]$$

$$= \frac{\cancel{\sin(0)} + e^{\rightarrow 1}}{-1 \cancel{e^{\cos(0)}}} = \boxed{-1} \text{ (Ans.)}$$

[Using
L'Hôpital's
Rule]

Differentiation

6

$$L = \lim_{x \rightarrow \infty} \frac{e^x}{e^{2x}} = 0 \quad \checkmark$$

$$= \lim_{x \rightarrow \infty} \frac{e^x}{2e^{2x}} = \frac{1}{2} \left(\lim_{x \rightarrow \infty} \frac{e^x}{e^{2x}} \right)$$

$$\therefore L = \frac{1}{2}L$$

$$\hookrightarrow \boxed{L=0} \quad \checkmark$$

TS1: L is defined
& finite

How??

Differentiation

Bartle-Sharbot
Ex 6.3
Q 5

7

$$f(x) = x^2 \sin\left(\frac{1}{x}\right) \quad \forall (x \neq 0)$$

$$g(x) = \sin(x)$$

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \frac{x^2 \sin\left(\frac{1}{x}\right)}{\sin(x)} = \lim_{x \rightarrow 0} \left[\cancel{\frac{x}{\sin x}} \left[\cancel{x \sin \frac{1}{x}} \right] \right]$$

1 0

$$\lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} \rightarrow \lim_{x \rightarrow 0} \frac{\left\{ 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \right\}}{1 \cdot \cos(x)}$$

lim doesn't exist!

Differentiation

Bartle-Sherbert
Ex 6.3
Q7c, 8a

8

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{\ln(\cos x)}{x} &= \lim_{x \rightarrow 0^+} \frac{1}{\cos x} \cdot (-\sin x) \quad \left[\text{Using L'Hôpital's Rule} \right] \\ &= \lim_{x \rightarrow 0^+} -\tan x = -\tan(0) \\ &= 0 \quad \checkmark \text{ (Ans.)}\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\tan^{-1}(x)}{x} \\ &= \lim_{x \rightarrow 0} \frac{\left(\frac{1}{1+x^2} \right)}{1} \quad \boxed{= 1} \quad \checkmark \text{ (Ans.)}\end{aligned}$$

Differentiation

Bartle - Snobart
Ex 6.3
Q 8b

9

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1}{x(\ln x)^2} &= \lim_{x \rightarrow 0} \frac{1/x}{(\ln x)^2} \\&= \lim_{x \rightarrow 0} \frac{-1/x^2}{2 \ln x \cdot \frac{1}{x}} = -\frac{1}{2} \lim_{x \rightarrow 0} \frac{1}{x \ln(x)} \\&= -\frac{1}{2} \lim_{x \rightarrow 0} \frac{1/x}{\ln(x)} = -\frac{1}{2} \lim_{x \rightarrow 0} \frac{-1/x^2}{1/x} \\&= \frac{1}{2} \lim_{x \rightarrow 0} \frac{1}{x} \boxed{= \infty} \quad (\because \underline{0 < x < 1}) \\&\quad \quad \quad \checkmark \quad (\text{Ans.})\end{aligned}$$

Differentiation

Bartle-Sharbat

Ex 53

Q 11 b, 11a

10

$$L = \lim_{x \rightarrow 0^+} (\sin x)^x$$

$$\ln L = \lim_{x \rightarrow 0^+} x \ln(\sin x)$$

$$\Delta = \lim_{x \rightarrow 0^+} \frac{\ln(\sin x)}{1/x} = \lim_{x \rightarrow 0^+} \frac{\cos x \cdot \frac{1}{\sin x}}{-1/x^2} = \lim_{x \rightarrow 0^+} \left(\frac{x}{\sin x} \right) x \cos x$$

$$\therefore \ln L = 0 \Rightarrow \boxed{L = 1} \checkmark (\text{Ans})$$

$$L = \lim_{x \rightarrow \frac{\pi}{2}^-} (\sec x - \tan x) = \lim_{x \rightarrow \frac{\pi}{2}^-} \left(\frac{1 - \sin x}{\cos x} \right) = \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{1 - \cos x}{1 + \sin x}$$

$$\boxed{= 0} \checkmark (\text{Ans})$$

Differentiation

Bartle - Sherbert

Ex 6.3

Q 12

11

$$\lim_{x \rightarrow \infty} (f(x) + f'(x)) = L$$

TST: $\lim_{x \rightarrow \infty} f(x) = L$ & $\lim_{x \rightarrow \infty} f'(x) = 0$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{e^x f(x)}{e^x} = \lim_{x \rightarrow \infty} \frac{e^x (f(x) + f'(x))}{\cancel{e^x}} = \underline{L}$$

This completes the proof! 

Differentiation

Bartle-Sharbat
Ex 6.3
Q14

12

$$\frac{d}{dn} x^n = x^n (1 + \ln x) \quad \text{for } \underline{\underline{x > 0}}$$

$$\lim_{x \rightarrow c} \frac{x^x - x^c}{x^x - c^c} = \lim_{x \rightarrow c} \frac{cx^{c-1} - x^c \ln x}{x^x (1 + \ln x)}$$

$$= \frac{cc^{c-1} - c^c \ln c}{c^c (1 + \ln c)}$$

$$= \frac{1 - \ln c}{1 + \ln c}$$



Differentiation

Rudin Ch5
Exc
6.3

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$$f(x) = x + \varepsilon g(x) \Rightarrow f'(x) = 1 + \varepsilon g'(x)$$

$$\Rightarrow 1 - \varepsilon M \leq f'(x) \leq 1 + \varepsilon M$$

($|g'(x)| \leq M$)

Choose $0 < \varepsilon < \frac{1}{2M}$

$$\Rightarrow 0 < \varepsilon M < \frac{1}{2} \Rightarrow \underline{\underline{\frac{1}{2} < 1 - \varepsilon M < 1 + \varepsilon M < \frac{3}{2}}}$$

From ① & ②:

$$\frac{1}{2} < f'(x) < \frac{3}{2} \Rightarrow f \text{ is strictly increasing} \Rightarrow \boxed{f \text{ is one-one if } \varepsilon < \frac{1}{2M}}$$

$\checkmark$ (true)

Differentiation

Rudin Ch-5

Exc

& 12

14

$$f(x) = |x|^3 = \begin{cases} x^3 & \text{if } x \geq 0 \\ -x^3 & \text{if } x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 3x^2, & x > 0 \\ 0, & x = 0 \\ -3x^2, & x < 0 \end{cases} \quad \text{(claim)}$$

Proof:

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \rightarrow 0} \pm \frac{x^3}{x} = 0 \quad \blacksquare_1$$

$$f''(x) = 6|x| \quad \forall x \in \mathbb{R} \quad \text{(claim)}$$

$$\text{Proof: } \lim_{x \rightarrow 0} \frac{(\pm 3x^2) - 0}{x - 0}$$

$$= 0 \quad \blacksquare_2$$

$$f''(x) = 6|x| \\ = 6|x|$$

$\therefore f^{(3)}(0)$ doesn't exist

 $\blacksquare_3$

Differentiation

Rudin Ch-5

Exc

& 11

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$$f''(x) = \lim_{h \rightarrow 0} \frac{f'(x + \frac{h}{2}) - f'(x - \frac{h}{2})}{h}$$

$$= \lim_{h \rightarrow 0} \lim_{k \rightarrow 0}$$

$$f(x + \frac{h}{2})$$

$$= \lim_{k \rightarrow 0} \frac{f(x + \frac{h}{2} + \frac{k}{2}) - f(x + \frac{h}{2} - \frac{k}{2})}{k}$$

$$\left[\frac{(f(x + \frac{h}{2} + \frac{k}{2}) - f(x + \frac{h}{2} - \frac{k}{2})) - (f(x - \frac{h}{2} + \frac{k}{2}) - f(x - \frac{h}{2} - \frac{k}{2}))}{hk} \right]$$

Let $k=h$  Existence of parent limit is being implicitly assumed.

$$f''(x) = \lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}$$



Coming up
with example

HW

Differentiation

Rudin Ch-5

Exc

& 22c

16

Let $x_1 \in [a, b]$

Let x be the fixed point (formal assumption)

$$|x_{n+1} - x| = |f(x_n) - f(x)| \quad \exists x_n \text{ b/w } x_n \neq x.$$

$$= |f(x_n)| \cdot |x_n - x| \leq A |x_n - x|$$

$$\leq A^2 |x_{n-1} - x| \dots \leq A^n |x_1 - x|$$

$$\text{Putting } n \rightarrow \infty, \because A < 1 \leq A^n(b-a)$$

$$|x_{n+1} - x| \rightarrow 0 \Rightarrow \boxed{x_n \rightarrow x} \quad \left(\begin{array}{l} \text{our formal} \\ \text{assumption} \\ \text{holds} \end{array} \right)$$